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ON THE POSSIBILITY OF MEASURING THE SHAPE AND ORBIT PARAMETERS OF THE MOON BY OPTICAL LOCATION

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SUMMARY

The energetic conditions of the optical (luminous) location of the Moon with the aid of ruby lasers are considered alongside of resolution by distance. It is shown that the low resolution conditioned by scattering of a light beam in the terrestrial atmosphere does not allow to measure the distance Earth—Moon with a precision sufficient for astrometric aims. It is shown that the difficulties connected with the insufficient power of contemporary lasers and with atmosphere scattering may be by-passed in the case of delivering to the Moon a special light reflector. The computation of light reflection parameters and the analysis of its location are performed. A preliminary mathematical analysis is given of the problem of measuring by the results of optical location of the mean distance Δ_0 to the Moon, of the "frontal" radius $\underline{r}$ of the Moon, of the parallax constant $\varkappa_0$ and of the equatorial radius r_0 of the Earth. The errors are estimated and it is shown that they are several times smaller than those with which these parameters have been known at present.

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The astronomic characteristics of the Moon's orbit have so far been determined by way of angular observation measurements of various points of the lunar surface for different positions of the Moon relative to the Earth. Since the angles measured are small, the separate measurements of such a kind differ by not too high a precision and, though they were conducted by numerous researchers during a prolonged time at which time large statistical material was accumulated, the basic parameters of

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*O VOZMOZHNOSTI IZMERENIYA PARAMETROV FIGURY I ORBITY LUNY METODOM OPTICHESKOY LOKATSII the lunar orbit are determined quite roughly. In particular, the mean distance between the mass centers of the Moon and of the Earth (mean orbit radius) is known with an error of ~3—4 km, and the eccentricity $\underline{e}$ of the orbit—with a precision to $\sim 1 \cdot 10^{-7}$.

The shape of the Moon was studied so far mainly by stereophotogrammetric methods, at the basis of which lies the measurement of the stereoscopic effect of a pair of two photographs of the Moon taken at different librations. The precision of measurement by these methods of the Moon's radius $\underline{r}$, directed toward the Earth ("frontal radius") constitutes $\pm$ 4—5 km [1, 2]. This is why the values of the Moon's elongation (difference between the "frontal" and the normal radii to it), obtained by various researchers, have a great disparity [3, 4]: Franz obtained 1.7, Ritter—7.0, Gusev—12.2, Gopmann—8.8 km (new processing of Franz's photographs). Therefore, the data on the shape of the Moon are quite imprecise.

The coordinates of the various points of the Moon relative to the Earth are computed with the utilization of Brown's theory of motion of the Moon [5]. Despite the perfection of this theory, the errors in the indicated coordinates may reach several kilometers, which is basically linked with the great errors in the above quantities Δ_0 e, r, entering into the theory as constants. The optical location of the Moon provides the possibility of measuring directly the distances to the surface of the Moon, and, by the same token, opens, on principle, an entirely new approach to the solution of that problem.

We shall consider below the problem of the relationship of distances, measured by optical method between the Earth and the Moon with the orbit and form parameters of the Moon, and we shall demonstrate that this method allows to determine the indicated parameters in a one-time experiment with a precision several times better than the one presently available.

This review is conducted in the assumption of application of optical quantum generators and other means, technically realizable at present.

1. SOME REMARKS ON THE CONDITIONS OF OPTICAL LOCATION OF THE MOON

The precise measurement of the distance between the Earth and the Moon by means of the pulse optical location is beset with a series of difficulties that may be basically subdivided into two groups.

Related to the first are the limitations conditioned by insufficiently high parameters of contemporary lasers, and, in the first place, by the low energy of the light pulse.

Let us consider the conditions of the Moon's location. The energy of the Moon-reflected light signal may be computed by the formula of the location range in which the fourth power of the distance is substituted by the square (all the light beam energy hits the target):

$$W_{refl} = W_0 \frac{S_T}{\pi R^2} \rho k_{transm} k_{rec} k^2_{atm}$$

Here W_{refl} is the energy of the reflected signal, W_0 is the energy of the dispatched signal, S_T is the area of the receiving device, R is the mean distance to the Moon, $\rho = 0.1$ is the albedo of the Moon, k_{transm} and k_{rec} are the loss factors, respectively in the transmitters and receivers. k_{atm} is the loss factor in the atmosphere.

For the mean zenithal angles under the standard atmospheric conditions $k_{atm} \approx 0.8$. In order to estimate W_{refl}, we shall assume $S_T = 5.30\,m^2$, $k_{transm} = 0.75$, $k_{rec} = 0.35$ [7] . We have

$$W_{refl} \approx 2 \cdot 10^{-19} W_0$$

The contemporary means of light registration allow to measure light signals containing from units to tens of quanta (depending upon the wavelength). For example, the quantum output of multislot photomultipliers attains ~4—5% in the infrared region ($\lambda \approx 7000$ A), which allows, as an average, to register every 20th—25th photon. Having assigned ourselves a minimum flux of ~100 photons (that is, ~4—5 photoelectrons at the output of FEU), which may be reliably registered, we shall determine with the aid of formula (1') the required laser's pulse energy

$$W_0 \approx 150 \text{ Joules}$$

In the first experiments on the optical location completed in 1962 [8] and in 1963 [7] , the then most powerful ruby lasers were utilized ($\lambda = 6943$ A) with 50 J energy in the pulse. In these two works the reflected signal constituted at the FEU output 1—1.5 photoelectrons, which the correctness of the above estimate of W_0 corroborates experimentally. At present lasers are worked out on ruby with energy in the pulse of the order of several hundred Joules [9] . The application of these lasers for the optical location of the Moon might have given the possibility of registering reliably every reflected pulse. However, these lasers, just as those applied in the works [6, 8] have a pulse duration τ of the order of milliseconds. This is the basic cause, not permitting the solution of the problem of distance measurement to the Moon, for the measurement error, corresponding to the indicated τ, constitutes >300 km. The application of the resonator's Q-factor modulation in the laser allows to lower the pulse duration to $\tau \approx 10^{-8}$ sec and to bring the error in the measurement of the distance to a value of a few meters. However, the application of modulated Q-factor entails the energy decrease in the pulse to a value of about a few Joules, which again leads to the requirement of applying a statistical method of reception. This complicates substantially the problem of distance variation, particularly if the latter varies with time, as this takes place in the case of the Moon.

Therefore, we see that the available lasers have parameters ruling out the measurement of the distance to the Moon by a single pulse.

Let us consider further another group of limitations that are more basic than the above mentioned difficulties relative to the apparatus.

The divergence of the light beam at the optical telescope outlet may be made of the order of several tens of fractions of a second. At passing through the terrestrial atmosphere the light beam undergoes brief deviations and widens as a result of multiple scattering on the atmosphere inhomogeneities. As it leaves the atmosphere, the beam's energy is included in a cone of which the aperture is of 2—3" in usual atmospheric conditions, even if it is strictly parallel at the outset. Therefore, in the regime of transition, the angular resolution of the installation is determined by atmosphere scattering; at heavenly body location from the Earth it cannot, in principle, be improved by comparison with the above indicated. The area on the Moon, corresponding to that divergence,has a linear dimension, normal to the ray, of $\sim$3.5—5 km.

If we used a beam with such a divergence for the location of uneven portions of the lunar surface, the error in the range (washing out of the reflected signal in time) will be determined by relative heights of the relief within the bounds of the luminous spot. Moreover, the Moon's sphericity will lead to additional washing out of the reflected pulse, so much the greater that the further parcel being located is farther from the center of the visible disk. For example, for a point of the lunar surface situated at 500 km from the center of the visible disk the error in range, conditioned only by the Moon's sphericity, attains 1—1.5 km.

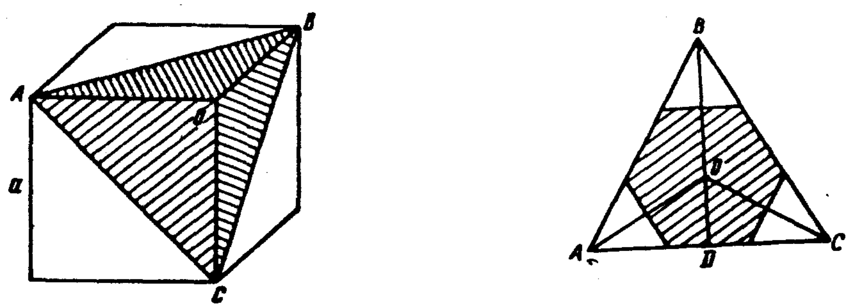
Therefore, the insufficient angular resolution conditioned by the terrestrial atmosphere, leads to the deterioration of the resolution by range, which limits considerably the scope of problems that may be resolved by way of optical location without involving additional technical means.

It is possible to broaden substantially the possibilities of optical location if an artificial light reflector is delivered on the Moon. With an appropriate choice of its parameters it is found to be possible to improve the energetic conditions of locations by one or several orders and thus practically fully avoid the limitations conditioned by the atmosphere scattering of the light beam.

2. ANALYSIS OF PARAMETERS OF AN ARTIFICIAL LIGHT REFLECTOR AND OF CONDITIONS OF ITS LOCATION

Let us examine how the energy of the reflected signal, having hit the receiving telescope on the ground, is related to the parameters of the corner (compensated) reflector.

A corner reflector is constituted of elements, each of which being a pyramid cut out of a cube by way of its cross section by the plane ABC (Fig. 1).



1a 1b

Fig. 1

Its volume is

$$V=\frac{a^3}{6}. \qquad (2)$$

For the case of normal incidence of rays on the plane ABC the reflecting area is that of the hexagon shaded in Fig. 1b and equal to

$$S=\frac{a^2}{\sqrt{3}}. \qquad (2')$$

When choosing the reflector parameters, its principal limitation is its weight. This is why, having, first of all, fixed the maximum reflector weight, we shall compute the number of sides required for obtaining the maximum signal reflection at the point of reception. The weight and the area of the angular reflector are

$$P_{\text{refl}}=V\delta N=\frac{a^3}{6}\delta N, \qquad S_{\text{refl}}=SN, \qquad (3)$$

where δ is the specific weight of the corner's material (we have in mind a cast side), and N is the number of sides (or elements). It may be seen from (3) that for a given P and a varying number of sides (elements) N, the edge a of the pyramid varies $\sim 1/\sqrt[3]{N}$, and the area of the entire reflector varies as $N/\sqrt[3]{N^2}=\sqrt[3]{N}$. The area illuminated by the signal on Earth, the real weight of the reflector being unchanged, is determined mainly by the diffraction divergence. We may disregard the ray's washing away on the return path through the atmosphere (it constitutes ~2" over 30 km of path). At rigorous calculation of the diffraction divergence of the reflector α_{refl}, it should be taken into account that the reflector aperture has a hexagonal shape. However, because of the impossibility of precise estimate of other quantities participating in the given calculation, we shall assume as a satisfactory approximation for the reflector aperture a circle of diameter

$$D=\frac{2a}{\sqrt{6}}. \tag{4}$$

inscribed in the hexagon.

If we consider that the faces of the corner reflector are disposed at random in the incident beam, it is clear that for a large number of faces such a reflector may be considered as a combination of spatially incoherent reflectors. The diffraction divergence of such a composed reflector maybe considered equal to that of a separate face. Therefore, the diffraction divergence of the reflected ray is taken equal to

$$\alpha_{refl}=2\frac{1{,}2\lambda}{D}=\frac{2{,}94\lambda}{a}. \tag{5}$$

The energy density at the reception point is

$$W_{rec} \sim \frac{S_{refl}}{S_M\, S_E}\ . \tag{6}$$

The area of the luminous spot on the Moon S_M is basically determined by the state of the atmosphere and is equal to

$$S_M = \pi\left(R\frac{\alpha_{atm}}{2}\right)^2,$$

where $\alpha_{atm}=3''$ is the angle of the light beam blow-up in the atmosphere, R is the distance to the Moon.

The area of the spot on Earth is

$$S_E = \pi\left(R\frac{\alpha_{отр}}{2}\right)^2=2{,}16\pi R^2\frac{\lambda^2}{a^2}. \tag{7}$$

For a given weight of the "corner" S_E changes with the variation of the number N as $\sqrt[3]{N^2}$. Consequentlv. the energy density at the point of reception is proportional to $N/\sqrt[3]{N^2}\sqrt[3]{N^2}=1/\sqrt[3]{N}$. with a variation of the number N. Hence, it follows that in order to obtain a maximum reflected signal it is necessary to apply a one-face reflector. Let us now express the value of the reflected signal as a function of the reflector's weight at N = 1 and compare it with the signal obtained without the reflector. At normal light beam incidence the density of the power flux on the corner reflector, without taking into account the losses is

$$w_{transm} = \frac{P_{transm}\ D_{transm}}{4\pi R^2} \qquad (8)$$

Here w_{transm} is the density of the power flux, P_{transm} is the power emitted by the laser. D_{transm} is the transmitting system's directivity factor; $D_{transm} = 4\pi R^2/S$. The density of the power flux, reflected from the corner reflector in the direction of the emission maximum at the receiving telescope, is

$$w_{rec} = \frac{w_{transm}\ S_{eff}}{4\pi R^2} \qquad (9)$$

where S_{eff} is the effective area of the corner reflector [10]:

$$S_{eff} = \frac{4\pi}{\lambda^2} S^2_{refl} = \frac{4\pi a^4}{3\lambda^2}.$$

The power received by the telescope P_{rec} is:

$$P_{rec} = w_{rec}\ S_T \qquad (10)$$

where S_T is the principal mirror's area of the telescope.

Therefore, the location formula of the range will be written in the form

$$P_{rec} = P_{transm} \frac{D_{transm}\ S_{eff} S_T}{(4\pi R^2)^2} \qquad (11)$$

In the case of oblique incidence the reflector area S_{ref} already decreases by about 20—30% at angles of 10—15°. The limitation to the precision of the reflector's orientation in the direction of the Earth is supplemented by it. Evidently it should not be worse than 15°. In order to take into account the inclined incidence, we shall introduce in formula (11) the factor $(0.7)^2 = 0.5$. The energy received by the telescope, taking into account the inclined incidence and the losses in the receiving and transmitting installations, will be written:

$$W_{rec} = W_{transm} \cdot \frac{0{,}5 a^4 S_T}{3\lambda^2 R^2 S_H} \cdot k_{transm} k_{rec} k^2_{atm} k_{refl} \cdot \qquad (12)$$

Here W_{transm} is the laser pulse energy, k_{transm} is the attenuation factor in the transmitting system, k_{rec} is the attenuation factor in the receiving system, k_{atm} is the attenuation factor in the atmosphere, k_{refl} is the reflection factor of the "corner." The energy received by the telescope in the case of diffusive reflection by the Moon's surface (without reflector) is determined by (1) . The gain in the reflected signal in the case of application of a reflector by comparison with the diffusive scattering, is

$$\beta = \frac{W_{rec}}{W'_{rec}} = 0{,}5\,\frac{\pi a^4 k_{refl.}}{3\lambda^2 \rho S_{n}} \tag{13}$$

Taking into account the possible contamination, we assume the value $k_{refl} \approx 0.5$. Substituting the numerical values into the relation (13), we shall obtain

$$\beta = 2{,}15\cdot 10^{-3} a^4 \approx 6{,}0\cdot 10^{-3} P^{4/3}. \tag{14}$$

Here <u>a</u> is expressed in cm, P in <u>g</u>. This formula is obtained for a glass reflector with $\delta = 2.7 \cdot 10^9 \cdot cm^{-3}$.

The diffraction divergence of the reflected beam by half-power will be in angular seconds

$$\alpha^{0.5}_{отр} = 1{,}02\,\frac{\lambda}{4{,}85\cdot 10^{-6} D} = \frac{18}{a} = \frac{14}{\sqrt[3]{P}}. \tag{15}$$

Compiled in the Table are the values of β, $\alpha^{0.5}_{refl}$ and <u>a</u> for several real values of weight.

P, кг	0,5	1,0	2,0	5,0	10,0
a, см	10,4	13,0	16,5	22,2	28,2
$\alpha^{0.5}_{отр}$ "	1,75	1,4	1,06	0,82	0,62
β	25,2	60	151	512	1330

It was shown above that for a reliable registration of a solitary signal diffusively scattered by the lunar surface, a laser energy of ~150 Joules is required. The available ruby lasers with modulated energy factor and in a regime of prolonged exploitation generate pulses with energies 4—5 J. In this case, for the registration of the response signal, it is necessary to apply in a single dispatching a reflector with a gain $\beta \approx 40$. The weight of such a reflector will constitute, according to formula (14), P = 0.72 k at $\alpha^{0.5}_{refl} = 1.54$.

Earlier it was assumed that the right angles between the edges of the reflector are maintained with absolute precision. In the presence of errors in the angles, the reception point may find itself on the side of the central diffraction maximum and the reflected signal will not be registered. It is obvious that a reliable reception of the reflected signal is possible on the condition that the aggregate beam deflection on account of errors in the angles between the edges of the reflector does not exceed the diffraction divergence by half-power. Taking into account the possiblity of the reception point's falling outside the diffraction maximum center, we must increase the gain by at least a factor of two by comparison with the above. For a reflector with $\beta = 80$, the weight $P = 1.18 \cdot 10^9$ kg, $\alpha^{0.5}_{refl} = 1.30''$ and $a = 13.9$ cm. It is evident that if the errors in the angles between the edges of the reflector constitute $\pm\Delta\alpha$, the reflected beam will be deflected from the initial direction by an angle of the order $\pm 6\,\Delta\alpha$. Thus, the condition for a reliable signal reception from a reflector with nonideal angles will be written in the form

$$12\Delta\alpha \leqslant \alpha^{0,5}_{refl.} \quad \text{or} \quad 6\Delta\alpha \leqslant \frac{\alpha^{0,5}_{refl.}}{2}. \tag{16}$$

For a reflector with $P = 1.18 \cdot 10^9$ kg, the errors in the angles between the edges must be no more than $\Delta\alpha \approx \pm 0'',1$.

Let us examine how the corner reflector parameters vary if the required precision is found to be practically unattainable. From formulas (15) and (16), we have

$$a' = \frac{1,5}{\Delta\alpha}, \quad P' = \frac{1,59}{\Delta\alpha^3}. \tag{17}$$

To obtain the former energy of the reflected signal, it is necessary to increase the number of units N which will lead to weight increase. Therefore, the errors in the angles compel one to select knowingly weight-wise disadvantageous reflector parameters. For an ideal, one-unit reflector, we have from formula (12)

$$\frac{W_{rec}}{W_{transm}} \sim a^4.$$

For a nonideal N-unit reflector (at great N), we have

$$\frac{W_{rec}}{W_{transm}} \sim N(a')^4 = N\left(\frac{1,5}{\Delta\alpha}\right)^4. \tag{18}$$

In order to obtain an identical gain β in both these cases, it is necessary that $N(1.5/\Delta\alpha) = a^4$ or

$$N = \frac{a^4\Delta\alpha^4}{3,38}.$$

Formulas (17) and (18), obtained with the application of the inequality (16), are valid only for a multi-unit reflector for which the aggregate error in the anlges, $12\,\Delta\alpha$ exceed $\alpha^{0.5}_{refl}$ of a single-unit reflector. Thus, a multi-unit reflector with the edge $a' = 1.5/\Delta\alpha$, one weight $P' = 1.59/\Delta\alpha^3$, and a number of units $N = a^4\Delta\alpha^4/3,38$. is the equivalent by the magnitude of the reflected signal to an ideal edge a and weight P. When computing a multi-unit reflector, the number N of units must be chosen sufficiently great so that statistical oscillations in the distribution of intensity over the diffraction picture that may appear on account of mutual interference between separate units may be neglected.

From the above said, it may be seen that the parameters of the reflector are sharply critical relative to the precision of the latter's preparation and also to the temperature and mechanical stability of its shape.

In the course of the preceding description, we did not touch upon the question of light background influence on the detection of the reflected signal. This question was considered in the works [6, 8]. The background of the dark part of the Moon, measured during direct experiments by optical location [6], constituted as an average $2 \cdot 10^3$ pulse/sec. The registration of the reflected signal against such a background requires the application of statistical methods of reception.

A substantial simplification of signal registration and the expansion of the possibilities of optical location may be achieved by applying the coincidence circuits on several FEU [11]. Thus, for example, the double-coincidence circuit allowed to locate a reflector located on the illuminated part of the lunar disk. For a reliable operation of such a circuit we have a reflector with a gain $\beta \approx 80$ and a laser with energy of the order of ~ 10 J. Under these conditions, the received signal consists of ~ 8 photoelectrons which is sufficient for working out the circuit with a probability close to the unity. The resolution time of the coincidence circuit should be chosen somewhat longer than the duration of the laser pulse with a modulating Q-factor, for example, of $2 \cdot 10^8$ sec. Let us estimate the probability of malfunctions of such a circuit when locating a reflector situated on the illuminated side of the Moon. Taking into account that the distance to the reflector on the Moon may be precalculated with a precision to ~ 4—5 km, for the registration of the reflected signal it is necessary to select a time interval of the order of 50 μsec duration, that is, within the limits of possible errors in depth. The orientation measurements of the background from the clear part of the Moon give in the spectral band 20 A about $5 \cdot 10^5$ pulse/sec to which correspond in the coincidence circuit $2.5 \cdot 10^3$ spurious pulses per second. This will constitute 0.12 spurious pulses for the chosen 50 μsec time interval. Moreover, possible also are malfunctions from the laser pulse reflected by the surface of the Moon. Depending upon the location of the reflector, the pulse may be washed out in depth from several hundred meters to 3—4 km on account of relief unevenness and Moon sphericity. At ~ 10 J energy in the pulse the signal from the lunar surface constitutes some 0.3 photoelectrons and is extended in time by about 1—10 μsec. In this case, the probability of malfunctions will constitute $4 \cdot 10^{-3} - 4 \cdot 10^{-4}$. The application of the triple-coincidence circuit will allow the location of the reflector in daytime. For a reliable performance of such a circuit, it is necessary to either increase the reflector gain to $\beta \approx 120$, or the laser energy to 15 joules which is attainable. In clear weather the sky background constitutes about $3 \cdot 10^6$ pulse/sec at a distance of 45° from the Sun. This results during registration time in $6 \cdot 10^{-2}$ spurious triple coincidences. Thus, the application of a coincidence circuit allows to conduct the optical (luminous) location of the Moon independently from its phase and the time of the day.

3. DETERMINATION OF THE MEAN DISTANCE TO THE MOON, OF THE "FRONTAL" RADIUS OF THE MOON, OF THE PARALLAX CONSTANT AND OF THE EQUATORIAL RADIUS OF THE EARTH BY THE RESULTS OF OPTICAL LOCATION

One of the direct results of optical location of a certain point of the surface of the Moon is the measurement of the distance to that point from the place of observation. This measured value may have no particular significance in itself, but, being related to a series of constants characterizing the dimensions and the shape of the Earth, the Moon, and the lunar orbit, it allows the refinement of some of them.

Let us examine which values are concrete and how they can be obtained from the processing of the results of optical location.

MEAN DISTANCE TO THE MOON AND THE "FRONTAL" RADIUS OF THE MOON

We represented in Fig. 2 a geometric scheme demonstrating the link between the measured distance D with certain parameters characterizing the shape of the Earth, of the Moon and of the lunar orbit. The distances Δ and Δ' and also the angles ω, γ, μ may be linked for the given moment of time with the coordinates of the points P and M by the following formulas

$$\Delta = \frac{a_0}{\sin \pi_{☾}}; \ \cos\gamma = \sin\varphi' \sin\delta + \cos\varphi' \cos\delta \cos t, \ \cos\omega = \sin b \sin b_0 + \cos b \cos b_0 \cos(l - l_0), \tag{19}$$
$$\Delta' = \sqrt{\Delta^2 + \rho^2 - 2\Delta\rho\cos\gamma}, \quad \sin\mu = \frac{r}{D}\sin\omega,$$

where a_0 is the equatorial radius of the Earth, $\delta, t, \pi_{☾}$ are the declination, the local hour angles and the parallax of the Moon, l_0 and b_0 are the libration of the Moon respectively by the longitude and the latitude taking into account the topocentric errors, l and b are the selenographic longitude and latitude of the point located. The measured distance D is linked with the quantities indicated in Fig. 2 by the following relation:

$$\cos\mu D = \Delta' - r\cos\omega = \sqrt{\Delta^2 + \rho^2 - 2\rho\Delta\cos\gamma} - r\cos\omega. \tag{20}$$

Let us borrow from the Brown theory of the Moon's motion the expansion of the Moon's parallax

$$\frac{a_0}{\Delta} = \frac{a_0}{\Delta_0}\{1 + 0{,}99272e\cos L + 0{,}18260\cos(L - 2D) + 0{,}98543e^2\cos 2L + 0{,}01642e\cos(L + 2D) + 0{,}0082\cos 2D + \ldots\}, \tag{21}$$

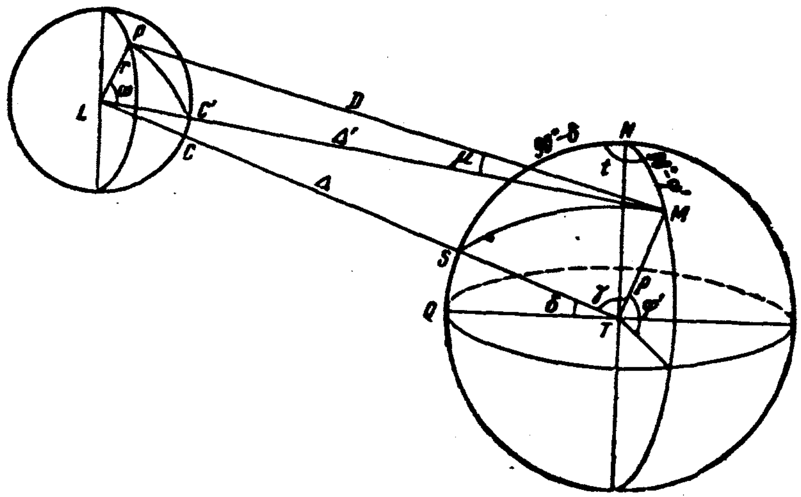


Fig. 2—Scheme showing the relationship between the measured distance D and the parameters of the system Earth-Moon

T is the Earth's mass center; L is the Moon's mass center; M is the point on the ground surface with geographic coordinates λ, φ' and the geocentric radius Q; P is the point located on the surface of the Moon with selenographic coordinates $\underline{l}$, $\underline{b}$, and selenocentric radius $\underline{r}$; $D = MP$, $\Delta' = ML$; $\Delta = TL$; γ is the geocentric angle between Q and Δ; μ is the topocentric angle between D and Δ' ; ω is the selenocentric angle between $\underline{r}$ and Δ'.

where Δ is the current distance between the mass center of the Earth and the Moon, Δ_0 is the mean distance between same, a_0 is the equatorial radius of the Earth, $\underline{e}$ is the ecentricity of the lunar orbit; L is the mean anomaly of the Moon, $\tilde{D}$ is the difference of the Moon's and Sun's mean longitudes. Substituting the expression for Δ from (21) into (20), we obtain an equation containing six unknowns: $\Delta_0, e, r, \rho, \omega, \gamma$. The determination of these parameters evidently amounts to the solution of a system of six equations of the form (20) written for six different points of the lunar orbit for which the distances D are measured. Let us examine how accurately the determination of the indicated parameters may be conducted by such a method. The solution of the system of equations (20) will be performed by the differential method allowing to make use of the approximate values of the quantities searched for and to find corrections for them.

Let D_c be the D computed by formulas (20) and (21) with known values of parameters entering in them obtained from the data of the Astronomical Yearbook. The value of the same distance obtained as a result of optical location will be D_0. The difference $D_0 - D_c = dD$ is conditioned by the difference between the values of the parameters $\Delta_0, r, \rho, \omega, e, \gamma$, known with limited precision, and their true values with measurement errors. In order to find the dependence between the corrections that must be introduced into the known values of parameters and the difference dD between the measured and the computed distances, we shall expand (20) in series. Limiting ourselves to terms of the first order relative to accretions, we shall obtain after certain transformations

$$\left(\frac{\Delta}{\Delta'} - \frac{\rho}{\Delta'}\cos\gamma\right)d\Delta - \cos\omega\, dr + \left(\frac{\rho}{\Delta'} - \frac{\Delta}{\Delta'}\cos\gamma\right)d\rho + r\sin\omega\, d\omega + + \frac{\Delta}{\Delta'}\rho\sin\gamma\, d\gamma = \cos\mu\, dD. \tag{22}$$

We may postulate with a precision to the unity of the fifth sign $\cos\mu = 1$ ($\mu_{max} \approx 16$) . Utilizing the expansion of the Moon's parallax (21), we shall substitute Δ_0 by $d\Delta_0$ and <u>de</u> Differentiating (21), we shall obtain

$$d\Delta = \frac{3422'',7}{\pi_{☾}}d\Delta_0 - \frac{3422'',7}{\pi_{☾}}\frac{a_0}{\sin\pi_{☾}}[0,99272\cos L + 0,18260\cos(L - 2\widetilde{D}) + + 0,10820\cos 2L + 0,01642\cos(L + 2\widetilde{D})]\, de. \tag{23}$$

Now substituting the expression (23) into (22) and denoting the series standing in (23) in square brackets by Q, we shall obtain an equation linking the corrections with the known values of the quantities with the correction to D in the final form

$$\left(\frac{\Delta}{\Delta'} - \frac{\rho}{\Delta'}\cos\gamma\right)\frac{3422'',7}{\pi_{☾}}d\Delta_0 - \cos\omega\, dr - \frac{3422'',7}{\pi_{☾}}\Delta Q\, de + + \left(\frac{\rho}{\Delta'} - \frac{\Delta}{\Delta'}\cos\gamma\right)d\rho + r\sin\omega\, d\omega + \frac{\Delta}{\Delta'}\rho\sin\gamma\, d\gamma = dD. \tag{24}$$

Analysis of the system of six equations of the form (20), written for six consecutive points of the lunar orbit in the course of one lunar month, shows that even at the most advantageous selection of the points the factors of the equations vary little from point to point (the orbit is nearly circular and the librations are small). This why the determinant of the system is small and the errors of the solutions, linked with the inaccuracy of measurement of the distance D, exceed significantly the errors, with which the parameters searched for are known or may be computed (for the estimates, the measurement error of the distance D is taken equal to $\varepsilon = \pm 3$ м).

In order to diminish the errors in the solutions, certain of the parameters searched for should be considered as well known quantities, utilizing to that effect their tabular or computed values with the respective errors. This introduces into the solutions additional errors but it also lowers the system's order and diminishes the errors linked with the measurement errors. Consideration of the system of equations, simplified by subsequent introduction of e, ρ, ω, γ as well known factors shows that the most advantageous in the sense of solution precision is the system of second order including two of the least precisely determined parameters r and Δ_0.

Let us consider a system of second order for additions dr and d Δ_0 to the parameters r and Δ_0 searched for

$$\left(\frac{\Delta_i}{\Delta_i'} - \frac{\rho}{\Delta_i'}\cos\gamma_i\right)\frac{3422'',7}{\pi_{☾}}d\Delta_0 - \cos\omega_i\, dr = dD_i\,(i = 1, 2). \qquad (25)$$

Here ω_i is the libration angle computed for every i. The corrections d Δ_0 and dr, determined from (25), will contain the following errors:

1) a random measurement error (enters into dD_i);
2) the systematic errors conditioned by the limited precision of parameters e, ρ, ω, γ, which are utilized for the determination of dD_i;
3) the systematic error on account of the imprecision with which the value of the speed of light is known.

Let us estimate these errors. Analyzing the system (25), it may be shown that most advantageous for measurements are the points of apogee and perigee. For these points the errors in d Δ_0 and dr only corresponding to the measurement error may be found as the solutions of the system of equations (25), in whose left-hand parts stand the numerical values of the factors for apogee and perigee, and in the right-hand parts those of ε_i:

$$\left(\frac{\Delta_1}{\Delta_1'} - \frac{\rho}{\Delta_1'}\gamma_1\right)\frac{3422'',7}{\pi_{☾_1}}(\delta\Delta_0)_\varepsilon - \cos\omega_1(\delta r)_\varepsilon = \varepsilon_1 \quad \text{for the apogee}$$

$$\left(\frac{\Delta_2}{\Delta_2'} - \frac{\rho}{\Delta_2'}\cos\gamma_2\right)\frac{3422'',7}{\pi_{☾_2}}(\delta\Delta_0)_\varepsilon - \cos\omega_2(\delta r)_\varepsilon = \varepsilon_2 \quad \text{for the perigee}$$

The determinant of this system is $\nabla \approx 0{,}1$ and the errors in dΔ_0 and dr (or in Δ_0 and r), determined by that method are $(\delta\Delta_0)_\varepsilon \approx (\delta r)_\varepsilon \approx (\varepsilon_1 + \varepsilon_2)/\nabla$. At a single measurement ε_i may be made $\sim \pm 3$ m, and the indicated statistical errors will constitute $(\delta\Delta_0)_\varepsilon \approx (\delta r)_\varepsilon \approx \pm 60$ м. The transformation of system (24) to the form (25) introduces into the solutions of $\tilde{d}\Delta_0$ and dr additional errors having the character of systematic errors so that $d\Delta_0 = \tilde{d}\Delta_0 + A$ and $dr = \tilde{d}r + B$, where d Δ_0 and dr are the solutions of the system (25), $\tilde{d}\,\Delta_0$ and $\tilde{d}r$ are solutions of the system (24), A and B are the referred to additional errors linked with the imprecision of e, ρ, ω and γ. Let us find A and B and, to that effect, let us subtract from each equation (25) the corresponding equation (24):

$$\left(\frac{\Delta_i}{\Delta_i'}-\frac{\rho}{\Delta_i'}\cos\gamma_i\right)\frac{3422'',7}{\pi_{☾_i}}A-\cos\omega_i B=-\frac{3422'',7}{\pi_{☾_i}}\Delta_i Q_i\,de+$$
$$+\left(\frac{\rho_i}{\Delta_i'}-\frac{\Delta_i}{\Delta_i'}\cos\gamma_i\right)d\rho+r\sin\omega_i\,d\omega+\frac{\Delta_i}{\Delta_i'}\rho\sin\gamma_i\,d\gamma\ (i=1,2). \qquad (26)$$

The quantities de, $d\rho$, $d\omega$ and $d\gamma$ in these equations represent the root-mean-square errors with which the quantities e, ρ, ω and γ are known or may be computed. Their numerical values are approximately as follows:

$$de\approx 1\cdot 10^{-7},\quad d\rho\approx\pm 60\ \text{м},\quad d\omega\approx\pm 40'',\quad d\gamma\approx\pm 0{,}1''.$$

The equations (26) with the numerical values of the factors for the points of apogee and perigee will be written as follows

$$1{,}05A-0{,}98B=4\cdot 10^5 de-0{,}5d\rho-1700(0\div 0{,}1)\,d\omega-6400\cdot 0{,}9d\gamma,$$
$$0{,}95A-1{,}00B=-4\cdot 10^5 de-0{,}6d\rho-1700\cdot 0{,}2\cdot d\omega-6400\cdot 0{,}6d\gamma.$$

This system is linear relative to de, $d\rho$, $d\omega$, $d\gamma$. We shall find its partial solutions A_m and B_m (where m is consecutively e, ρ, ω and γ). corresponding separately to the root-mean-square errors de, $d\rho$, $d\omega$ and $d\gamma$. The determinant of such a system is $\nabla\approx 0{,}1$, and the factors of the equations are near 1. This is why $A_m\approx B_m$. The numerical values of the partial solutions A_m and B_m are as follows

$$A_e\approx B_e\approx\pm 800,\ A_\rho\approx B_\rho\approx\pm 60,\ A_\omega\approx B_\omega\approx\pm 30^1,\ A_\gamma\approx B_\gamma\approx\pm 40\ \text{м}.$$

The errors in Δ_0 and r, linked with the error in the value of the speed of light $\delta c\approx\pm 10^{-6}$ are determined analogously. They have a value $(\delta\Delta_0)_c\approx\pm 300$ м.

Inasmuch as all the above computed errors are independent root-mean-square errors, the total error in Δ_0 and r is

$$\delta\Delta_0=\sqrt{(\delta\Delta_0)_s^2+\sum_m A_m^2+(\delta\Delta_0)_c^2}\approx\pm 860\ \text{м},$$
$$\delta r=\sqrt{(\delta r)_s^2+\sum_m B_m^2+(\delta r)_c^2}\approx\pm 810\ \text{м}.$$

---

* For the error to be small, it is necessary that the point located be situated near the center of the Moon's disk and at the moments of apogee and perigee time. This is possible at the time when the line of lunar orbit nodes is near the line of apsides. We assumed that the point being located is at the center of the disk with a precision 0.5° ($\sin\omega\approx 0{,}01$).

EQUATORIAL RADIUS OF THE EARTH AND THE LUNAR PARALLAX CONSTANT

The correction obtained according to the results of optical location allows to refine the values of the Earth's radius and of the lunar parallax constant. The mean distance to the Moon is linked with the equatorial radius of the Earth a_0 and the mean parallax of the Moon by the relation

$$\sin \pi_0 = \frac{a_0}{\Delta_0}. \tag{27}$$

We shall utilize the expression of the dynamic parallax of the Moon [12]

$$\sin^2 \pi_0 = \left(\frac{\pi_{☾} \sin 1''}{86\,400}\right)^2 \frac{a_0}{g_0} \frac{1+\sigma}{1+m} (1+\delta)^3. \tag{28}$$

From the relations (27) and (28), we may determine at a known Δ_0 the values of π_0 and a_0 provided we consider as accurately known the remaining values in the expression (28). We shall represent (27) and (28) in the differential form

$$d\pi_0 = \frac{1}{\cos \pi} \frac{da_0}{\Delta_0} - \frac{a_0}{\Delta_0} \frac{d\Delta_0}{\Delta_0} \frac{1}{\cos \pi} = \frac{1}{\cos \pi} \frac{da_0}{\Delta_0} - \operatorname{tg} \pi_0 \frac{d\Delta_0}{\Delta_0}, \tag{29}$$

$$d\pi_0 = \frac{1}{3} \operatorname{tg} \pi_0 \frac{da_0}{a_0}. \tag{30}$$

From (29) and (30) we shall obtain

$$\delta a_0 = -\frac{3}{2} \sin \pi \, \delta\Delta_0, \tag{31}$$

$$\delta \pi_0 = -\frac{5}{2} \operatorname{tg} \pi \frac{\delta\Delta_0}{\Delta_0}. \tag{32}$$

Substituting the numerical values of the factors, we shall finally obtain the expressions for the determination of corrections to the admitted values of the equatorial radius of the Earth and of the mean parallax of the Moon

$$\delta a_0 = -0{,}0249\ \delta\Delta_0 \approx \pm 20\ \text{м},\ \delta\pi_0 = -0{,}02226\ [\delta\Delta_0\ \text{км}] \approx \pm 0{,}015. \tag{33}$$

RADIUS VECTOR OF THE POINT OF OBSERVATION

Let us consider (24) for the two moments of time in the course of one night when the Moon is in upper culmination and has a maximum zenithal distance possible for observations. Let us simplify the system (24)

$$\left(\frac{\Delta_i}{\Delta_i'} - \frac{\rho}{\Delta_i'} \cos \gamma\right) \frac{3422'',7}{\pi_{☾_i}} d\Delta_0 + \left(\frac{\rho}{\Delta_i'} - \frac{\Delta_i}{\Delta_i'} \cos \gamma_i\right) d\rho = dD_i \, (i = 1, 2).$$

The estimate of the correction error $d\rho$ will be conducted analogously to the estimate of errors $d\Delta_0$ and dr. The determinant of the system (33) will be mainly dependent upon the difference in angles γ_1, γ_2 and may be made equal to ~ 0.7. The error in $d\rho$ on account of the imprecision of measurements is

$$(\delta\rho)_\varepsilon \approx \pm\frac{\varepsilon\sqrt{2}}{0{,}7} \approx \pm 6 \text{ м},$$

where — $\varepsilon \approx 3$ м is the root-mean-square error of the distance measurement. The bringing of the system (24) up to the system (33) introduces into the solution additional errors:

$$d\rho = \widetilde{d\rho} + R, \quad d\Delta_0 = \widetilde{d\Delta_0} + A,$$

where, as in previous estimates of errors $d\Delta_0$ and dr, $d\Delta_0$ and $d\rho$ are the solutions of the system (33), $\widetilde{d\Delta_0}, \widetilde{d\rho}$ are the solutions of the system (24), R and A and the above referred to additional errors. The system for R and A has the form

$$\left(\frac{\Delta_i}{\Delta_i'} - \frac{\rho}{\Delta_i'}\cos\gamma_i\right)\frac{2422'',7}{\pi_{☾_i}} A + \left(\frac{\rho}{\Delta_i'} - \frac{\Delta_i}{\Delta_i'}\cos\gamma_i\right) R = -\cos\omega_i\, dr +$$

$$+ \frac{3422'',7}{\pi_{☾_i}} \Delta_i Q_i\, de - r\sin\omega_i\, d\omega - \frac{\Delta_i}{\Delta_i'}\rho\sin\gamma\, d\gamma \quad (i = 1, 2).$$

At the condition that $\delta r \approx 1$ км, $\omega_1 - \omega_2 \approx 0°{,}5$ and $d\gamma \approx 0'',1$, we shall obtain

$$R_r \approx \pm 4 \text{ м}, \quad R_e \approx \pm 3 \text{ м}, \quad R_\omega \approx \pm 8 \text{ м}, \quad R_\gamma \approx \pm 6 \text{ м}.$$

The total error in ρ is $\delta\rho = \sqrt{R_r^2 + R_e^2 + R_\omega^2 + R_\gamma^2 + (\delta\rho)_\varepsilon^2} \approx \pm 15$ м.

Thus, the analysis conducted allows to draw the conclusion that the basic parameters of the orbit and of the shape of the Moon as well as the Earth's equatorial radius may be refined by the optical location method. Note that the mathematical consideration of the question presented here is far from being complete. A more detailed investigation, which is now being conducted, shows that at a special selection of observation conditions it is possible to increase still more substantially the precision in the determination of the investigated parameters.

*** THE END ***

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